

# THE SELF-SIMILAR AND MULTIFRACTAL NATURE OF A NETWORK TRAFFIC MODEL

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**ABSTRACT.** We look at a family of models for Internet traffic with increasing input rates and consider approximation models which exhibit self-similarity at large time scales and multifractality at small time scales. Depending on whether the input rate is fast or slow, the total cumulative input traffic can be approximated by a self-similar stable Lévy motion or a self-similar Gaussian process. The stable Lévy limit does not depend on the behavior of the individual transmission schedules but the Gaussian limit does. Also, the models and their approximations show multifractal behavior at small time scales.

## 1. INTRODUCTION

Empirical research over the last decade has firmly established that data network models should incorporate concepts like self-similarity, heavy tails and long-range dependence. Using an empirical study done at Bellcore, Leland et al. [1994] were first to give evidence in the context of the LAN traffic. Further discussions can be obtained from Erramilli et al. [1996], Resnick [1997, 2001], Willinger et al. [1995]. These studies were mostly concerned with large time scale behavior of the traffic at scales above a few hundred milliseconds. Recently, empirical studies have focussed on traffic at scales of a few hundred milliseconds or lower. The WAN traffic at small time scales show complicated multifractal behavior, as has been observed empirically in Mannersalo and Norros [1997], Paxson and Floyd [1995], Riedi and Lévy Véhel [1997]. Also in a recent paper, Baccelli and Hong [2002] observed multifractal behavior for traffic generated by the simulation of TCP-type network traffic. These observations stimulated researchers to look for a model which could explain both the microscopic as well as the macroscopic behaviors. In Feldman et al. [1998a], Feldmann et al. [1998b], Gilbert et al. [1998, 1999], Kulkarni et al. [2001], Riedi and Willinger [2000], attempts have been made to consider a conservative cascade model as the transmission schedule. A different approach was taken in Maulik and Resnick [2002] where the  $M/G/\infty$  input model was considered. The resulting multifractal behavior of the aggregated cumulative input traffic process at small time scale is explained by the multifractality of the individual transmission schedules. This explanation has recently been justified empirically by Baccelli and Hong [2002] for simulated traces of TCP traffic. On the other hand, for large time scales, Maulik and Resnick [2002] obtained a stable Lévy motion as a large time scale approximation.

Though, the model proposed by Maulik and Resnick [2002] succeeded in integrating the empirically observed behaviors for the two time scales, the result for large time scales was not completely satisfactory. The stable Lévy limit in that paper does not depend on the distributional behavior of the individual transmission schedules. Also due to lack of empirical evidence for heavy tailed traffic rates, a Gaussian limit would be desirable. Riedi and Willinger [2000] have argued in favor of a Gaussian approximation in their work.

Mikosch et al. [2002] considered a family of  $M/G/\infty$  models with increasing input rate and showed that the possible limits for large time scales depend on the growth of the input rate and

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may be either fractional Brownian motion (fBm) or a stable Lévy motion. However, they considered a deterministic linear transmission schedule, which is unrealistic and does not allow for multifractal behavior at small time scales.

The current paper integrates the above strains of research. We propose a sequence of  $M/G/\infty$  models along the lines of Mikosch et al. [2002] with random transmission schedules. The models incorporate multifractal behavior at the microscopic level. At the macroscopic level, for the slow growth of the input rate, we get a stable Lévy limit, whereas the fast growth condition gives a Gaussian limit, which depends on the self-similarity index of the individual transmission schedules.

This paper is arranged as follows: Section 2 gathers the notation used in this paper. Section 3 describes the family of models and discusses the critical input rates. In Sections 4 and 5, we collect useful results needed for further analyses. In Section 6 and 7, we consider the slow and the fast growth cases respectively.

## 2. NOTATION

We need to introduce the following notation for our discussion. For a non-decreasing function  $x$ , we define its left continuous inverse as

$$(2.1) \quad x^{\leftarrow}(t) = \inf\{u : x(u) \geq t\}.$$

For a non-negative random variable  $U$ , we denote its distribution function by  $F_U$ , i.e.,  $F_U(u) = \mathbb{P}[U \leq u]$ . Let  $\bar{F}_U(u) = 1 - F_U(u)$ . Then we define the function  $\tilde{b}_U$  as

$$\tilde{b}_U(T) = \inf\left\{u : \bar{F}_U(u) \leq \frac{1}{T}\right\} = \left(\frac{1}{\bar{F}_U}\right)^{\leftarrow}(T).$$

Recall that a function  $\phi$  is *regularly varying* of index  $\alpha$  and is denoted by  $\phi \in RV_\alpha$  (cf. Section 0.4 of Resnick [1987]), if for all  $u > 0$ ,

$$\lim_{t \rightarrow \infty} \frac{\phi(tu)}{\phi(t)} = u^\alpha.$$

We say that  $U$  has a tail of index  $\alpha_U > 0$ , if  $\bar{F}_U \in RV_{-\alpha_U}$ . In such a case (cf. Proposition 0.8(v) of Resnick [1987]),  $\tilde{b}_U \in RV_{\alpha_U^{-1}}$  and also we have

$$(2.2) \quad \lim_{T \rightarrow \infty} T \mathbb{P}[U_1 > \tilde{b}_U(T)u] = u^{-\alpha_U}.$$

Conversely, if (2.2) holds, then  $\tilde{b}_U \in RV_{\alpha_U^{-1}}$  and  $\bar{F}_U \in RV_{-\alpha_U}$ . In either case, we can choose a strictly increasing, absolutely continuous function  $b_U$ , such that  $\tilde{b}_U \sim b_U$ , i.e.,

$$\lim_{T \rightarrow \infty} \frac{\tilde{b}_U(T)}{b_U(T)} = 1$$

(cf. Proposition 0.8(vii) of Resnick [1987]). We can further say that

$$(2.3) \quad \lim_{T \rightarrow \infty} T \mathbb{P}[U_1 > b_U(T)u] = u^{-\alpha_U}.$$

We call this function  $b_U$  the quantile function of the non-negative random variable  $U$ .

### 3. THE MODEL

For the purpose of obtaining both the Gaussian and stable limits, we consider a family of models indexed by  $T \in \mathbb{R}_+$ , along the lines considered by Mikosch et al. [2002]. For the  $T$ -th model:

- (1) We denote the time when  $k$ -th transmission begins by  $\Gamma_k^{(T)}$ . The sequence  $\{\Gamma_k^{(T)}\}$  is non-negative and strictly increasing to  $\infty$ .
- (2) The size of the file transmitted is  $J_k$  and we assume  $J_k > 0$ .
- (3) The transmission schedule is denoted by  $A_k(\cdot)$ , where  $A_k(t)$  denotes the amount of data that can be transmitted in time  $t$  after the  $k$ th transmission has begun. It is a non-decreasing càdlàg function starting at 0 and increasing to  $\infty$ , which vanishes on the negative real axis. It ensures that a file of any finite size can be transmitted in finite time.

The quantity of interest is the cumulative input traffic defined as

$$(3.1) \quad X^{(T)}(t) = \sum_{k=1}^{\infty} A_k \left( t - \Gamma_k^{(T)} \right) \wedge J_k.$$

The length of  $k$ -th transmission is defined as

$$L_k = \inf\{t : A_k(t) \geq J_k\} = A_k^-(J_k).$$

$F_L$  denotes the marginal distribution of the transmission lengths, and satisfies

$$F_L(x) = P[L_1 \leq x] = P[A_1(x) \geq J_1].$$

**3.1. Small time scale behavior.** To study the behavior of the cumulative traffic process  $X^{(T)}(\cdot)$  for small time scales, we need to make the following further minimal assumptions on the transmission schedule  $\{A_k\}$ :

- (4) We assume  $\{A_k\}$  are identically distributed and have stationary increments.
- (5) The multifractal spectrum of  $A_k(\cdot)$  is not degenerated to a single point, which ensures that we consider processes with paths that show real multifractal behavior.
- (6) The multifractal spectrum of  $A_k(\cdot)$  restricted to any (non-random) interval is non-random.

For the definitions and discussion regarding the multifractal spectrum, we invite the reader to consult Maulik and Resnick [2002, Section 3.3] and Riedi [2001].

*Remark 3.1.* If  $A_k$  is, for example, an increasing Lévy process, then, restricted to any interval, it has a non-random multifractal spectrum for the Hölder exponent based on exponential growth rate [cf. Jaffard, 1999, Maulik and Resnick, 2002, Section 3.5].

In Maulik and Resnick [2002, Section 5], it has been shown, using a path-by-path analysis, that under the assumptions (1)-(6), for each  $T$ , the multifractal spectrum of the process  $X^{(T)}$  coincides with that of  $A_1$  almost everywhere.

**3.2. Large time scale behavior.** For large time scale analysis, we need to make distributional assumptions, which are:

- (7) We assume  $\{\Gamma_k^{(T)}, k \geq 1\}$  form a homogeneous Poisson process with intensity parameter  $\lambda(T)$ , called the *input rate*. We assume  $\lambda(T)$  to be non-decreasing.
- (8) We assume  $\{A_k, k \geq 1\}$  and  $\{J_k, k \geq 1\}$  are independent of each other and are i.i.d. sequences independent of  $\{\Gamma_k^{(T)}, k \geq 1\}$ .
- (9) We assume the tail of the distribution of  $J_1$  is regularly varying of index of  $-\alpha_J$ , where  $\alpha_J \in (1, 2)$ , i.e.,  $\bar{F}_J \in RV_{-\alpha_J}$ . Hence  $J_1$  has finite first moment denoted by  $\mu_J$ .

(10) The transmission schedule  $A_1$  is  $H$ -self-similar ( $H$ -ss), where  $H$  satisfies:

- (a)  $H\alpha_J > 1$ .
- (b)  $H < \frac{1}{\alpha_J - 1}$ .
- (c)  $H < \frac{1}{2 - \alpha_J}$ .

These conditions guarantee that  $H$  can neither be too large or too small.

(11) We also put the following moment conditions on  $A_1$ :

$$\mathbb{E} [A_1(1)^{-\alpha_J}] < \infty \text{ and } \mathbb{E} [A_1(1)^{2-\alpha_J+\delta}] < \infty,$$

for some  $\delta > 0$ . Further uses of  $\delta$  assume assumption (11) holds.

*Remark 3.2.* Assumptions (4)-(6) are path by path and are required for small time scale analysis. The microscopic analysis does not require any distributional assumptions on the transmission initiation times. When (4)-(6) as well as (7)-(11) hold, the result about the multifractal spectrum of each  $X^{(T)}(\cdot)$  holds as well as the large time scale approximations to be described.

Assume further that  $A_1$  has stationary increments. Since  $A_1$  has increasing paths and is  $H$ -ss with stationary increments which is not identically zero, from Theorem 2.1 of Vervaat [1994], we must have  $H \geq 1$ . When  $H = 1$ , we have  $A_1(t) \equiv A_1(1)t$  almost surely. But this leads to the case where paths are linear and hence not multifractal. So we consider only the case  $H > 1$ . Then  $A_1(1)$  is a positive stable random variable of index  $\frac{1}{H}$  and hence has a density which decays exponentially near 0 (cf. Theorem 2.5.2 of Zolotarev [1986]) and so has all negative moments finite. Also  $A_1(1)$  has all positive moments smaller than  $1/H$  finite. Hence, assumption (10c) guarantees the moment condition (11). Also Remark 3.1 shows that a Lévy process satisfies the conditions for the multifractal analysis. Thus a  $\frac{1}{H}$ -stable Lévy process can be made to satisfy all the requirements on transmission schedules.

*Remark 3.3.* From Proposition 7.2 of Maulik and Resnick [2002], assumptions (8), (9) and (11) imply that  $\bar{F}_L \in RV_{-H\alpha_J}$ , and hence by assumption (10a),  $L$  has finite mean denoted by  $\mu_L$ . Actually, a closer look at the proof of Proposition 7.2 of Maulik and Resnick [2002] further gives us that

$$(3.2) \quad \bar{F}_L(T) \sim \bar{F}_J(T^H)$$

If we also assume that  $L_1$  has infinite variance, i.e.,  $H\alpha_J < 2$ , then  $\alpha_J < 2$  implies

$$\frac{\alpha_J}{\alpha_J - 1} = 1 + \frac{1}{\alpha_J - 1} > 2 > H\alpha_J,$$

and so assumption (10b) holds.

**3.3. Critical input rate.** The results of large time scale analysis depend on the input rate. Depending on whether the input rate is *slow* or *fast* - a concept made precise in the following - we can get either a stable or a Gaussian limit.

The input rate is called slow, if

$$(S) \quad \lim_{T \rightarrow \infty} \frac{b_J(\lambda(T)T)}{T^H} = 0,$$

and it is called fast, if

$$(F) \quad \lim_{T \rightarrow \infty} \frac{b_J(\lambda(T)T)}{T^H} = \infty.$$

The following lemmas provide alternate approaches to the above conditions.

**Lemma 3.1.** *The slow growth condition (S) is equivalent to*

$$(3.3) \quad \lim_{T \rightarrow \infty} \lambda(T)T\bar{F}_L(T) = 0.$$

On the other hand, the fast growth condition (F) is equivalent to

$$(3.4) \quad \lim_{T \rightarrow \infty} \lambda(T)T\bar{F}_L(T) = \infty.$$

*Proof.* First we prove the conditions (S) and (F) imply (3.3) and (3.4) respectively. We define  $0 < \varepsilon(T) := b_J(\lambda(T)T)/T^H$  and we have  $T^H\varepsilon(T) \rightarrow \infty$ , since  $b_J(\lambda(T)T) \rightarrow \infty$ . Then, using (3.2)

$$\lambda(T)T\bar{F}_L(T) \sim \lambda(T)T\bar{F}_J(T^H) \sim \frac{\bar{F}_J(T^H)}{\bar{F}_J(T^H\varepsilon(T))},$$

which converges to 0 or  $\infty$  according as the condition (S) or (F) holds, since  $\bar{F}_J \in RV_{-\alpha_J}$ .

Conversely, define

$$\tilde{\varepsilon}(T) := \lambda(T)T\bar{F}_L(T) \sim \lambda(T)T\bar{F}_J(T^H).$$

Then

$$\frac{b_J(\lambda(T)T)}{T^H} \sim \frac{b_J(\tilde{\varepsilon}(T)/FJbar(T^H))}{b_J(1/FJbar(T^H))},$$

which converges to 0 or  $\infty$  according as  $\tilde{\varepsilon}(T)$  goes to 0 or  $\infty$ , since  $b_J \in RV_{1/\alpha_J}$ .  $\square$

The following lemma gives implications of the growth conditions which are useful when applying the growth conditions.

**Lemma 3.2.** *The slow growth condition (S) implies*

$$(3.5) \quad \lim_{T \rightarrow \infty} \frac{\lambda(T)T^{H+1}\bar{F}_L(T)}{b_J(\lambda(T)T)} = 0.$$

*The limit is  $\infty$  when condition (F) holds.*

*Proof.* Define  $\varepsilon(T) = b_J(\lambda(T)T)/T^H$  as before. Then  $\lambda(T)T \sim 1/\bar{F}_J(T^H\varepsilon(T))$ . Thus,

$$\frac{\lambda(T)T^{H+1}\bar{F}_L(T)}{b_J(\lambda(T)T)} \sim \frac{\lambda(T)T^{H+1}\bar{F}_J(T^H)}{b_J(\lambda(T)T)} \sim \frac{\bar{F}_J(T^H)}{\varepsilon(T)\bar{F}_J(T^H\varepsilon(T))} = \frac{T^H\bar{F}_J(T^H)}{T^H\varepsilon(T)\bar{F}_J(T^H\varepsilon(T))},$$

which goes to 0 or  $\infty$  according as the condition (S) or (F) holds, since  $s\bar{F}_J(s) \in RV_{1-\alpha_J}$ .  $\square$

#### 4. DECOMPOSITION

Our analysis requires the Poisson point process

$$M^{(T)} := \sum_{k=1}^{\infty} \varepsilon_{\left(\Gamma_k^{(T)}, A_k, J_k\right)},$$

which has mean measure  $\lambda(T)d\gamma \times P[A_1 \in da] \times F_J(dj)$  on  $(0, \infty) \times \mathbb{D}_{\uparrow} \times (0, \infty)$ , where  $\mathbb{D}_{\uparrow}$  is the space of non-decreasing càdlàg functions on  $(0, \infty)$ . The random variable  $X^{(T)}(t)$  is the following function of  $M$  restricted to  $\mathcal{R}(t) = \{(\gamma, a, j) \in (0, \infty) \times \mathbb{D}_{\uparrow} \times (0, \infty) : \gamma < t\}$ :

$$X^{(T)}(t) = \sum_{k=1}^{\infty} \left[ A_k \left( t - \Gamma_k^{(T)} \right) \wedge J_k \right] \mathbf{1}_{\mathcal{R}(t)} \left( \Gamma_k^{(T)}, A_k, J_k \right) = \iiint_{(\gamma, a, j) \in \mathcal{R}(t)} (a(t - \gamma) \wedge j) M(d\gamma, da, dj).$$

It helps to split  $\mathcal{R}(t)$  in two disjoint sets

$$\mathcal{R}_1(t) = \{(\gamma, a, j) \in (0, \infty) \times \mathbb{D}_{\uparrow} \times (0, \infty) : \gamma < t, j \leq a(t - \gamma)\}$$

and

$$\mathcal{R}_2(t) = \{(\gamma, a, j) \in (0, \infty) \times \mathbb{D}_{\uparrow} \times (0, \infty) : \gamma < t, j > a(t - \gamma)\}.$$

$\mathcal{R}_1(t)$  and  $\mathcal{R}_2(t)$  correspond to the regions where transmission has ended or is continuing respectively, by time  $t$ . Correspondingly, the input process  $X^{(T)}$  breaks into two sums:

$$(4.1) \quad X_1^{(T)}(t) = \sum_{k=1}^{\infty} J_k \mathbf{1}_{\mathcal{R}_1(t)} \left( \Gamma_k^{(T)}, A_k, J_k \right)$$

and

$$(4.2) \quad X_2^{(T)}(t) = \sum_{k=1}^{\infty} A_k \left( t - \Gamma_k^{(T)} \right) \mathbf{1}_{\mathcal{R}_2(t)} \left( \Gamma_k^{(T)}, A_k, J_k \right).$$

Since  $X_i^{(T)}(t)$ ,  $i = 1, 2$  are functions of  $M^{(T)}|_{\mathcal{R}_i(t)}$ ,  $i = 1, 2$  respectively with  $\mathcal{R}_1(t) \cap \mathcal{R}_2(t) = \emptyset$ , we have  $X_1^{(T)}(t)$  and  $X_2^{(T)}(t)$  are independent.

For any  $t > 0$ , we also observe the following facts about the regions  $\mathcal{R}_i(Tt)$ ,  $i = 1, 2$ , as  $T \rightarrow \infty$ :

$$(4.3) \quad m_1(Tt) =: \frac{1}{\lambda(T)} \mathbb{E} \left[ M^{(T)}(\mathcal{R}_1(Tt)) \right] = \int_{\gamma=0}^{Tt} \mathbb{P}[L_1 \leq Tt - \gamma] d\gamma =: \hat{F}_L(Tt) \sim Tt,$$

and

$$(4.4) \quad m_2(Tt) =: \frac{1}{\lambda(T)} \mathbb{E} \left[ M^{(T)}(\mathcal{R}_2(Tt)) \right] = \int_{\gamma=0}^{Tt} \mathbb{P}[L_1 > Tt - \gamma] d\gamma \sim \mu_L.$$

So the mean measure restricted to  $\mathcal{R}_1(Tt)$  or  $\mathcal{R}_2(Tt)$  is finite for fixed  $T, t$  and we can have the Poisson representations

$$(4.5) \quad M_i^{(T)}|_{\mathcal{R}_i(Tt)} \stackrel{d}{=} \sum_{k=1}^{P_i^{(T)}} \varepsilon_{(\tau_{k,i}^{(T)}, S_{k,i}^{(T)}, W_{k,i}^{(T)})},$$

where  $P_i^{(T)}$  is a Poisson random variable with parameter  $\lambda(T)m_i(Tt)$ , which is independent of the i.i.d. sequence  $\left\{ (\tau_{k,i}^{(T)}, S_{k,i}^{(T)}, W_{k,i}^{(T)}) : k \in \mathbb{N} \right\}$  with common joint distribution

$$(4.6) \quad \frac{d\gamma \mathbb{P}[A_1 \in d\alpha] F_J(dw)}{m_i(Tt)} \Big|_{\mathcal{R}_i(Tt)}.$$

Note  $t$  is fixed in this argument and is sometimes suppressed in the notations for the sake of brevity. Also we define random variables  $(\tau_i^{(T)}, S_i^{(T)}, W_i^{(T)})$  independent of  $P_i^{(T)}$  and distributed identically as  $(\tau_{1,i}^{(T)}, S_{1,i}^{(T)}, W_{1,i}^{(T)})$ , for  $i = 1, 2$ . Then we can rewrite  $X_i^{(T)}(Tt)$ ,  $i = 1, 2$  in terms of the above Poisson representation (4.5) as:

$$(4.7) \quad X_1^{(T)}(Tt) \stackrel{d}{=} \sum_{k=1}^{P_1^{(T)}} W_{k,1}^{(T)},$$

$$(4.8) \quad X_2^{(T)}(Tt) \stackrel{d}{=} \sum_{k=1}^{P_2^{(T)}} S_{k,2}^{(T)} (Tt - \tau_{k,2}^{(T)}).$$

## 5. MOMENT BEHAVIOR

We will need moments of the above summands. We first consider the following modest variant of Potter's bound [cf. Potter, 1940, Resnick, 1987] for regularly varying functions.

**Lemma 5.1.** *Let  $\phi$  be a regularly varying function of index  $\alpha > -1$ . Then given  $\varepsilon > 0$  and  $x_0 > 0$ , there exists  $T_0$  such that for  $T > T_0$ , and  $x > x_0$ , we have*

$$(1 - \varepsilon)\phi(T)x^{\alpha+1-\varepsilon} < (\alpha + 1) \int_0^x \phi(Tu)du < (1 + \varepsilon)\phi(T)x^{\alpha+1+\varepsilon}.$$

*Proof.* The result follows from Potter's bounds applied to  $\int_0^x \phi(u)du \in RV_{1+\alpha}$ .  $\square$

An analogous result can easily be proved for regularly varying functions  $\phi$  of index smaller than  $-1$ , which is summarized in the following lemma:

**Lemma 5.2.** *Let  $\phi$  be a regularly varying function of index  $\alpha < -1$ . Then given  $\varepsilon > 0$  and  $x_0 > 0$ , there exists  $T_0$  such that for  $T > T_0$ , and  $x > x_0$ , we have*

$$(1 - \varepsilon)\phi(T)x^{\alpha+1-\varepsilon} < -(\alpha + 1) \int_x^\infty \phi(Tu)du < (1 + \varepsilon)\phi(T)x^{\alpha+1+\varepsilon}.$$

Now, we study the moments of the summands in the representations (4.7) and (4.8). Recall from assumption (11), we have  $E[A_1(1)^{2-\alpha_J+\delta}] < \infty$ . Then, for  $\alpha_J < l < 2 + \delta/2$ , we have since  $A_1$  is  $H$ -ss,

$$\begin{aligned} E \left[ \left( W_1^{(T)} \right)^l \right] &\sim \frac{1}{Tt} \int_{\gamma=0}^{Tt} \int_{w=0}^{\infty} lw^{l-1} P[w < J_1 \leq A_1(Tt - \gamma)] dw d\gamma \\ &= (Tt)^{Hl} \bar{F}_J((Tt)^H) \int_{\gamma=0}^1 \int_{w=0}^{\infty} lw^{l-1} \frac{1}{\bar{F}_J((Tt)^H)} P \left[ w < \frac{J_1}{(Tt)^H} \leq A_1(\gamma) \right] dw d\gamma \\ (5.1) \quad &= (Tt)^{Hl} \bar{F}_J((Tt)^H) \int_{\gamma=0}^1 E \left[ \int_{w=0}^{\infty} lw^{l-1} \frac{\bar{F}_J((Tt)^H w) - \bar{F}_J((Tt)^H A_1(\gamma))}{\bar{F}_J((Tt)^H)} \mathbf{1}_{[w < A_1(\gamma)]} dw \right] d\gamma. \end{aligned}$$

Now, the integrand on the right side of (5.1) is bounded by

$$(5.2) \quad lw^{l-1} \frac{\bar{F}_J((Tt)^H w)}{\bar{F}_J((Tt)^H)} [\mathbf{1}_{[A_1(1) > w \vee 1]} + \mathbf{1}_{[w \leq 1]}] \rightarrow lw^{l-\alpha_J-1} [\mathbf{1}_{[A_1(1) > w \vee 1]} + \mathbf{1}_{[w \leq 1]}]$$

as  $T \rightarrow \infty$  and

$$\begin{aligned} &\int_{\gamma=0}^1 E \left[ \int_{w=0}^{\infty} lw^{l-1} \frac{\bar{F}_J((Tt)^H w)}{\bar{F}_J((Tt)^H)} [\mathbf{1}_{[A_1(1) > w \vee 1]} + \mathbf{1}_{[w \leq 1]}] dw \right] d\gamma \\ (5.3) \quad &= E \left[ \int_{w=0}^{A_1(1)} lw^{l-1} \frac{\bar{F}_J((Tt)^H w)}{\bar{F}_J((Tt)^H)} \mathbf{1}_{[A_1(1) > 1]} dw \right] + \int_{w=0}^1 lw^{l-1} \frac{\bar{F}_J((Tt)^H w)}{\bar{F}_J((Tt)^H)} dw. \end{aligned}$$

Now, by Karamata's theorem, the second term on the right side of (5.3) converges to

$$(5.4) \quad \frac{l}{l - \alpha_J} = \int_{\gamma=0}^1 \mathbb{E} \left[ \int_{w=0}^{\infty} l w^{l-\alpha_J-1} \mathbf{1}_{[w \leq 1]} dw \right] d\gamma.$$

For the first term on the right side of (5.3), observe that  $T^{l-1} \bar{F}_J(T)$  is regularly varying with index  $l - \alpha_J - 1 > -1$ , and hence by the upper bound from Lemma 5.1, there is a  $T_0$ , which is non-random, such that for all  $T > T_0$ , we have

$$\int_{w=0}^{A_1(1)} l w^{l-1} \frac{\bar{F}_J((Tt)^H w)}{\bar{F}_J((Tt)^H)} \mathbf{1}_{[A_1(1) > 1]} dw < \left(1 + \frac{\delta}{2}\right) \frac{l}{l - \alpha_J} A_1(1)^{l+\frac{\delta}{2}-\alpha_J} \mathbf{1}_{[A_1(1) > 1]},$$

which is integrable by assumption (11). Then, by the Dominated Convergence Theorem, the first term on the right side of (5.3) converges to

$$\mathbb{E} \left[ \frac{l}{l - \alpha_J} A_1(1)^{l-\alpha_J} \mathbf{1}_{[A_1(1) > 1]} \right] = \int_{\gamma=0}^1 \mathbb{E} \left[ \int_{w=0}^{\infty} l w^{l-\alpha_J-1} \mathbf{1}_{[A_1(1) > w \vee 1]} dw \right] d\gamma,$$

which along with (5.4) shows

$$(5.5) \quad \begin{aligned} & \int_{\gamma=0}^1 \mathbb{E} \left[ \int_{w=0}^{\infty} l w^{l-1} \frac{\bar{F}_J((Tt)^H w)}{\bar{F}_J((Tt)^H)} [\mathbf{1}_{[A_1(1) > w \vee 1]} + \mathbf{1}_{[w \leq 1]}] dw \right] d\gamma \\ & \rightarrow \int_{\gamma=0}^1 \mathbb{E} \left[ \int_{w=0}^{\infty} l w^{l-\alpha_J-1} [\mathbf{1}_{[A_1(1) > w \vee 1]} + \mathbf{1}_{[w \leq 1]}] dw \right] d\gamma \end{aligned}$$

Then, from (5.2) and (5.5), using Pratt's lemma [cf. Pratt, 1960, Resnick, 1998], we are allowed to take the limit under the integral sign in the right side of (5.1) to get, for  $\alpha_J < l < 2 + \delta/2$ ,

$$(5.6) \quad \begin{aligned} \mathbb{E} \left[ \left( W_1^{(T)} \right)^l \right] & \sim T^{Hl} \bar{F}_J(T^H) t^{H(l-\alpha_J)} \int_{\gamma=0}^1 \mathbb{E} \left[ \int_{w=0}^{\infty} l w^{l-1} [w^{-\alpha_J} - A_1(\gamma)^{-\alpha_J}]_+ dw \right] d\gamma \\ & = T^{Hl} \bar{F}_J(T^H) t^{H(l-\alpha_J)} \int_{\gamma=0}^1 \mathbb{E} \left[ \int_{w=0}^{A_1(\gamma)} l w^{l-1} [w^{-\alpha_J} - A_1(\gamma)^{-\alpha_J}] dw \right] d\gamma \\ & = \frac{\alpha_J}{l - \alpha_J} T^{Hl} \bar{F}_J(T^H) t^{H(l-\alpha_J)} \int_{\gamma=0}^1 \mathbb{E} [A_1(\gamma)^{l-\alpha_J}] d\gamma \\ & = \frac{\alpha_J}{(l - \alpha_J)[H(l - \alpha_J) + 1]} T^{Hl} \bar{F}_J(T^H) t^{H(l-\alpha_J)} \mathbb{E} [A_1(1)^{l-\alpha_J}] \\ & \sim \frac{\alpha_J}{(l - \alpha_J)[H(l - \alpha_J) + 1]} T^{Hl} \bar{F}_L(T) t^{H(l-\alpha_J)} \mathbb{E} [A_1(1)^{l-\alpha_J}]. \end{aligned}$$



Also, using monotone convergence, we observe that as  $T \rightarrow \infty$ ,

$$(5.7) \quad \mathbb{E} \left[ W_1^{(T)} \right] \sim \int_{\gamma=0}^1 \int_{w=0}^{\infty} \mathbb{P} [w < J_1, L_1 \leq Tt\gamma] dw d\gamma \uparrow \mu_J.$$

For future reference, note from (5.6) and (5.7) that

$$(5.8) \quad \lim_{T \rightarrow \infty} \frac{\text{Var} \left[ W_1^{(T)} \right]}{T^{2H} \bar{F}_L(T)} = \frac{\alpha_J}{(2 - \alpha_J)[H(2 - \alpha_J) + 1]} t^{H(2 - \alpha_J)} \mathbb{E} [A_1(1)^{2 - \alpha_J}] =: \sigma_1^2 t^{H(2 - \alpha_J)}$$

and

$$(5.9) \quad \limsup_{T \rightarrow \infty} \frac{\mathbb{E} \left[ \left| W_1^{(T)} - \mathbb{E} [W_1^{(T)}] \right|^{2 + \frac{\delta}{2}} \right]}{T^{H(2 + \frac{\delta}{2})} \bar{F}_L(T)} \leq \limsup_{T \rightarrow \infty} \frac{\mathbb{E} \left[ \left( W_1^{(T)} \right)^{2 + \frac{\delta}{2}} \right] + \left( \mathbb{E} [W_1^{(T)}] \right)^{2 + \frac{\delta}{2}}}{T^{H(2 + \frac{\delta}{2})} \bar{F}_L(T)},$$

which is a finite constant.

Next we study the moments of the summands of (4.8). First we consider the moments of order  $l$ , with  $\alpha_J < l < 2 + \delta/2$ . We have, using self-similarity of  $A_1$  and (4.4), that

$$\begin{aligned} \mathbb{E} \left[ S_2^{(T)} \left( Tt - \tau_2^{(T)} \right) \right]^l &\sim \frac{1}{\mu_L} \int_{\gamma=0}^{Tt} \int_{w=0}^{\infty} lw^{l-1} \mathbb{P} [w < A_1(Tt - \gamma) < J_1] dw d\gamma \\ &= \frac{1}{\mu_L} (Tt)^{Hl+1} \int_{\gamma=0}^1 \int_{w=0}^{\infty} lw^{l-1} \mathbb{P} \left[ w < \frac{A_1(Tt\gamma)}{(Tt)^H} < \frac{J_1}{(Tt)^H} \right] dw d\gamma \\ &= \frac{1}{\mu_L} (Tt)^{Hl+1} \bar{F}_J((Tt)^H) \int_{\gamma=0}^1 \int_{w=0}^{\infty} lw^{l-1} \frac{1}{\bar{F}_J((Tt)^H)} \mathbb{P} \left[ w < A_1(\gamma) < \frac{J_1}{(Tt)^H} \right] dw d\gamma \\ (5.10) \quad &= \frac{1}{\mu_L} (Tt)^{Hl+1} \bar{F}_J((Tt)^H) \int_{\gamma=0}^1 \mathbb{E} \int_{w=0}^{\infty} lw^{l-1} \frac{\bar{F}_J((Tt)^H A_1(\gamma))}{\bar{F}_J((Tt)^H)} \mathbf{1}_{[A_1(\gamma) > w]} dw d\gamma, \end{aligned}$$

and bounding the integrand above as in the case of  $l$ -th moments of  $W_1^{(T)}$  with  $\alpha_J < l < 2 + \delta/2$ , we justify the interchange of the limit and integral to obtain

$$\begin{aligned} \mathbb{E} \left[ S_2^{(T)} \left( Tt - \tau_2^{(T)} \right) \right]^l &\sim \frac{1}{\mu_L} T^{Hl+1} \bar{F}_L(T) t^{H(l - \alpha_J) + 1} \int_{\gamma=0}^1 \mathbb{E} [A_1(\gamma)^{l - \alpha_J}] d\gamma \\ (5.11) \quad &= \frac{1}{\mu_L [H(l - \alpha_J) + 1]} T^{Hl+1} \bar{F}_L(T) t^{H(l - \alpha_J) + 1} \mathbb{E} [A_1(1)^{l - \alpha_J}]. \end{aligned}$$

The first moment requires more careful analysis in this case. As for the higher moments, we again have, using self-similarity of  $A_1$ , (see (5.10)),

$$\mathbb{E} \left[ S_2^{(T)} \left( Tt - \tau_2^{(T)} \right) \right]$$

$$\begin{aligned}
& \sim \frac{1}{\mu_L} (Tt)^{H+1} \bar{F}_J((Tt)^H) \int_{\gamma=0}^1 \mathbb{E} \left[ \int_{w=0}^{\infty} \frac{\bar{F}_J((Tt)^H A_1(\gamma))}{\bar{F}_J((Tt)^H)} \mathbf{1}_{[A_1(\gamma) > w]} dw \right] d\gamma \\
& = \frac{1}{\mu_L} (Tt)^{H+1} \bar{F}_J((Tt)^H) \int_{\gamma=0}^1 \mathbb{E} \left[ \int_{w=0}^{\infty} \frac{\bar{F}_J((Tt)^H A_1(1)\gamma^H)}{\bar{F}_J((Tt)^H)} \mathbf{1}_{[A_1(1)\gamma^H > w]} dw \right] d\gamma \\
& = \frac{1}{\mu_L} (Tt)^{H+1} \bar{F}_J((Tt)^H) \int_{\gamma=0}^1 \mathbb{E} \left[ A_1(1)\gamma^H \frac{((Tt)^H A_1(1)\gamma^H)}{\bar{F}_J((Tt)^H)} \right] d\gamma \\
(5.12) \quad & = \frac{1}{\mu_L} (Tt)^{H+1} \bar{F}_J((Tt)^H) \frac{1}{H} \mathbb{E} \left[ A_1(1)^{-\frac{1}{H}} \int_{\nu=0}^{A_1(1)} \frac{Tt\nu^{\frac{1}{H}} \bar{F}_J((Tt)^H \nu)}{Tt\bar{F}_J((Tt)^H)} d\nu \right],
\end{aligned}$$

where we substitute  $\nu = A_1(1)\gamma^H$  in the last step. Now, by Karamata's theorem, as  $T \rightarrow \infty$ ,

$$(5.13) \quad \int_{\nu=0}^{A_1(1)} \frac{Tt\nu^{\frac{1}{H}} \bar{F}_J((Tt)^H \nu)}{Tt\bar{F}_J((Tt)^H)} d\nu \rightarrow \frac{A_1(1)^{\frac{1}{H}-\alpha_J+1}}{\frac{1}{H}-\alpha_J+1},$$

since by assumption (10b)  $H < \frac{1}{\alpha_J-1}$ . Also, there exists a non-random  $T_0$ , such that for all  $T > T_0$ , we have,

$$\begin{aligned}
\int_{\nu=0}^{A_1(1)} \frac{Tt\nu^{\frac{1}{H}} \bar{F}_J((Tt)^H \nu)}{Tt\bar{F}_J((Tt)^H)} d\nu & \leq \int_{\nu=0}^1 \frac{Tt\nu^{\frac{1}{H}} \bar{F}_J((Tt)^H \nu)}{Tt\bar{F}_J((Tt)^H)} d\nu + \int_{\nu=0}^{A_1(1)} \frac{Tt\nu^{\frac{1}{H}} \bar{F}_J((Tt)^H \nu)}{Tt\bar{F}_J((Tt)^H)} d\nu \mathbf{1}_{[A_1(1) > 1]} \\
& < \frac{1 + \frac{\alpha_J-1}{2}}{\frac{1}{H} - (\alpha_J-1)} \left[ 1 + A_1(1)^{\frac{1}{H}-\frac{\alpha_J-1}{2}} \mathbf{1}_{[A_1(1) > 1]} \right],
\end{aligned}$$

where we bound the first term using Karamata's theorem and the second term using the upper bound from Lemma 5.1. Thus,

$$A_1(1)^{-\frac{1}{H}} \int_{\nu=0}^{A_1(1)} \frac{Tt\nu^{\frac{1}{H}} \bar{F}_J((Tt)^H \nu)}{Tt\bar{F}_J((Tt)^H)} d\nu < \frac{\alpha_J+1}{2(\frac{1}{H} - (\alpha_J-1))} \left[ A_1(1)^{-\frac{1}{H}} + 1 \right],$$

which is integrable, since  $\mathbb{E}[A_1(1)^{-\alpha_J}] < \infty$  by assumption (11) and  $1/H < \alpha_J$ . So by the Dominated Convergence Theorem and (5.13), we have from (5.12),

$$(5.14) \quad \mathbb{E} \left[ S_2^{(T)} \left( Tt - \tau_2^{(T)} \right) \right] \sim \frac{1}{\mu_L} T^{H+1} \bar{F}_L(T) t^{1-H(\alpha_J-1)} \frac{\mathbb{E}[A_1(1)^{1-\alpha_J}]}{1-H(\alpha_J-1)}.$$

For future reference we collect some results about the variance and other centered moments using (5.14) and (5.11),

$$\begin{aligned}
(5.15) \quad \lim_{T \rightarrow \infty} \frac{\text{Var} \left[ S_2^{(T)} \left( Tt - \tau_2^{(T)} \right) \right]}{T^{2H+1} \bar{F}_L(T)} & = \frac{1}{\mu_L [H(2-\alpha_J)+1]} t^{H(2-\alpha_J)+1} \mathbb{E} [A_1(1)^{2-\alpha_J}] \\
& =: \frac{1}{\mu_L} \sigma_2^2 t^{H(2-\alpha_J)+1}
\end{aligned}$$

and

$$\begin{aligned}
 (5.16) \quad & \limsup_{T \rightarrow \infty} \frac{\mathbb{E} \left[ \left| S_2^{(T)}(Tt - \tau_2^{(T)}) - \mathbb{E} \left[ S_2^{(T)}(Tt - \tau_2^{(T)}) \right] \right|^{2+\frac{\delta}{2}} \right]}{T^{H(2+\frac{\delta}{2})+1} \bar{F}_L(T)} \\
 & \leq \limsup_{T \rightarrow \infty} \frac{\mathbb{E} \left[ \left( S_2^{(T)}(Tt - \tau_2^{(T)}) \right)^{2+\frac{\delta}{2}} \right] + \left( \mathbb{E} \left[ S_2^{(T)}(Tt - \tau_2^{(T)}) \right] \right)^{2+\frac{\delta}{2}}}{T^{H(2+\frac{\delta}{2})+1} \bar{F}_L(T)},
 \end{aligned}$$

which is a finite constant.

## 6. HEAVY TAILED APPROXIMATION UNDER THE SLOW GROWTH CONDITION

To study implications of the slow growth condition, we need to look at the tail behavior of  $W_1^{(T)}$ .

**Proposition 6.1.** *Under the slow growth condition (S), the random variable  $W_1^{(T)}$  given in (4.5) and (4.6) satisfies*

$$(6.1) \quad \lim_{T \rightarrow \infty} \lambda(T) T \mathbb{P} \left[ W_1^{(T)} > b_J(\lambda(T)T) w \right] = w^{-\alpha_J}.$$

*Proof.* Observe that

$$\begin{aligned}
 (6.2) \quad \lambda(T) T \mathbb{P} \left[ W_1^{(T)} > b_J(\lambda(T)T) w \right] & \sim \frac{\lambda(T)T}{Tt} \int_0^{Tt} \mathbb{P} [b_J(\lambda(T)T) w < J_1 \leq A_1(\gamma)] d\gamma \\
 & \leq \lambda(T) T \bar{F}_J(b_J(\lambda(T)T) w) \rightarrow w^{-\alpha_J}.
 \end{aligned}$$

Hence,

$$(6.3) \quad \limsup_{T \rightarrow \infty} \lambda(T) T \mathbb{P} \left[ W_1^{(T)} > b_J(\lambda(T)T) w \right] \leq w^{-\alpha_J}.$$

On the other hand, from (6.2),

$$\begin{aligned}
 (6.4) \quad & \lambda(T) T \mathbb{P} \left[ W_1^{(T)} > b_J(\lambda(T)T) w \right] \\
 & \sim \frac{(b_J(\lambda(T)T))^{\frac{1}{H}}}{T} \int_0^{\frac{T}{(b_J(\lambda(T)T))^{\frac{1}{H}}}} \lambda(T) T \mathbb{P} \left[ b_J(\lambda(T)T) w < J_1 \leq A_1 \left( \gamma (b_J(\lambda(T)T))^{\frac{1}{H}} t \right) \right] d\gamma \\
 & \geq \frac{(b_J(\lambda(T)T))^{\frac{1}{H}}}{T} \int_N^{\frac{T}{(b_J(\lambda(T)T))^{\frac{1}{H}}}} \lambda(T) T \mathbb{P} \left[ b_J(\lambda(T)T) w < J_1 \leq A_1 \left( N (b_J(\lambda(T)T))^{\frac{1}{H}} t \right) \right] d\gamma \\
 & = \left( 1 - N \frac{(b_J(\lambda(T)T))^{\frac{1}{H}}}{T} \right) \lambda(T) T \mathbb{P} \left[ b_J(\lambda(T)T) w < J_1 \leq A_1 \left( N (b_J(\lambda(T)T))^{\frac{1}{H}} t \right) \right] \\
 (6.5) \quad & \sim \lambda(T) T \mathbb{P} \left[ b_J(\lambda(T)T) w < J_1 \leq A_1 \left( N (b_J(\lambda(T)T))^{\frac{1}{H}} t \right) \right] \\
 (6.6) \quad & = \lambda(T) T \mathbb{P} \left[ w < \frac{J_1}{b_J(\lambda(T)T)} \leq A_1(Nt) \right]
 \end{aligned}$$

$$(6.7) \quad \rightarrow \mathbb{E}[w^{-\alpha_J} - (A_1(Nt))^{-\alpha_J}]_+,$$

where the inequality (6.4) holds for any natural number  $N$  for sufficiently large  $T$ , since, by the slow growth condition (S), we have  $b_J(\lambda(T)T)/T^H \rightarrow 0$ . The equivalence (6.5) holds for the same reason. The equality in (6.6) follows from the  $H$ -self-similarity of  $A_1$ . Finally the convergence (6.7) holds by the regular variation of the tail of  $J_1$  and Dominated Convergence Theorem. Then letting  $N$  go to  $\infty$ , using the fact  $A_1(\infty) = \infty$  and Dominated Convergence Theorem, we have

$$(6.8) \quad \liminf_{T \rightarrow \infty} \lambda(T)T \mathbb{P} \left[ W_1^{(T)} > b_J(\lambda(T)T)w \right] \geq w^{-\alpha_J}.$$

The inequalities (6.3) and (6.8) together complete the proof.  $\square$

Using the tail behavior in the above Proposition 6.1 and an analysis based on the point process as in, for example, Maulik et al. [2000, Section 5], we can conclude that

$$(6.9) \quad \frac{\sum_{k=1}^{P_1^{(T)}} W_{k,1}^{(T)}}{b_J(\lambda(T)T)} - \frac{P_1^{(T)} \mathbb{E} [W_1^{(T)}]}{b_J(\lambda(T)T)} \Rightarrow Z_{\alpha_J}(t),$$

where  $Z_\alpha$  is  $\alpha$ -stable Lévy motion with mean 0, skewness 1 and scale  $C_\alpha^{\frac{1}{\alpha}}$  and

$$C_\alpha = \frac{1 - \alpha}{\Gamma(2 - \alpha) \cos\left(\frac{\pi\alpha}{2}\right)}.$$

Now observe that

$$(6.10) \quad \begin{aligned} & \frac{P_1^{(T)} \mathbb{E} [W_1^{(T)}] - \lambda(T)Tt\mu_J}{b_J(\lambda(T)T)} \\ &= \frac{P_1^{(T)} - \lambda(T)\hat{F}_L(Tt)}{\sqrt{\lambda(T)\hat{F}_L(Tt)}} \sqrt{\lambda(T)\hat{F}_L(Tt)} \frac{\mathbb{E} [W_1^{(T)}]}{b_J(\lambda(T)T)} - \frac{\lambda(T)}{b_J(\lambda(T)T)} \left( Tt\mu_J - \hat{F}_L(Tt) \mathbb{E} [W_1^{(T)}] \right). \end{aligned}$$

Since we know from (5.7) and the fact that  $\alpha_J < 2$  that

$$\sqrt{\lambda(T)\hat{F}_L(Tt)} \frac{\mathbb{E} [W_1^{(T)}]}{b_J(\lambda(T)T)} \sim \frac{\sqrt{\lambda(T)T}}{b_J(\lambda(T)T)} \sqrt{t}\mu_J \rightarrow 0,$$

and  $P_1^{(T)}$  is a Poisson random variable with mean  $\lambda(T)\hat{F}_L(Tt)$ , which goes to  $\infty$ , the first term on the right side of (6.10) is probabilistically negligible. As for the second term, observe that

$$\begin{aligned} & \frac{\lambda(T)}{b_J(\lambda(T)T)} \left( Tt\mu_J - \hat{F}_L(Tt) \mathbb{E} [W_1^{(T)}] \right) \\ &= \frac{\lambda(T)}{b_J(\lambda(T)T)} \int_{j=0}^{\infty} \int_{\gamma=0}^{Tt} \mathbb{P}[J_1 > j, L_1 > \gamma] d\gamma dj \\ &\leq \frac{\lambda(T)}{b_J(\lambda(T)T)} \left[ \int_{\gamma=0}^{Tt} \gamma^H \bar{F}_L(\gamma) d\gamma + \int_{\gamma=0}^{Tt} \int_{j=\gamma^H}^{\infty} \bar{F}_J(j) dj d\gamma \right] \end{aligned}$$

$$\sim \text{constant} \frac{\lambda(T)T^{H+1}\bar{F}_L(T)}{b_J(\lambda(T)T)} \rightarrow 0,$$

where the equivalence follows using Karamata's theorem, the fact  $1 + H(1 - \alpha_J) > 0$  from assumption (10b) and the relation (3.2) between the tails of the distributions of  $L_1$  and  $J_1$  and the limit holds because of (3.5). Thus the left side of (6.10) is probabilistically negligible. This fact combined with the convergence (6.9) gives us the weak convergence for the contribution of the first region under slow growth: for all  $t > 0$ ,

$$(6.11) \quad \frac{X_1^{(T)}(Tt) - \lambda(T)T\mu_J t}{b_J(\lambda(T)T)} \Rightarrow Z_{\alpha_J}(t).$$

For the contribution of the second region, observe from the representation (4.8) and the equivalence (5.14), we have

$$\frac{\mathbb{E}[X_2^{(T)}(Tt)]}{b_J(\lambda(T)T)} = \frac{\mathbb{E}[P_2^{(T)}] \mathbb{E}[S_2^{(T)}(Tt - \tau_2^{(T)})]}{b_J(\lambda(T)T)} \sim \text{constant} \frac{\lambda(T)T^{H+1}\bar{F}_L(T)}{b_J(\lambda(T)T)} \rightarrow 0,$$

where the limit follows from (3.5). Hence  $X_2^{(T)}(t)/(b_J(\lambda(T)T))$  is probabilistically negligible. Combining this fact with (6.11), we get for all  $t > 0$ ,

$$\frac{X^{(T)}(Tt) - \lambda(T)Tt}{b_J(\lambda(T)T)} \Rightarrow Z_{\alpha_J}(t).$$

We can check the finite dimensional convergence as in Maulik et al. [2002]. Thus, we have

**Theorem 6.1.** *Under the assumptions (7) - (11) and slow growth condition (S), we have,*

$$\frac{X^{(T)}(T\cdot) - \lambda(T)T\mu_J \cdot}{b_J(\lambda(T)T)} \xrightarrow{\text{fdi}} Z_{\alpha_J}(\cdot),$$

where the convergence is in the sense of weak convergence of finite dimensional distributions and  $Z_\alpha$  is  $\alpha$ -stable Lévy motion with mean 0, skewness 1 and scale  $C_\alpha^{\frac{1}{\alpha}}$ .

## 7. ASYMPTOTIC NORMALITY UNDER THE FAST GROWTH CONDITION

**7.1. One-dimensional convergence.** We use the moment conditions (5.8), (5.9), (5.15), (5.16) along with Lyapunov's Central Limit Theorem to study the behavior of  $X^{(T)}(Tt)$  under the fast growth condition (F). The fast growth condition (F) implies

$$(7.1) \quad \eta(T) = \sqrt{\lambda(T)T\bar{F}_L(T)} \rightarrow \infty$$

using (3.4).

Using (5.8) and (5.9) we get that

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \frac{[\lambda(T)Tt] \mathbb{E} \left[ \left| W_1^{(T)} - \mathbb{E}[W_1^{(T)}] \right|^{2+\frac{\delta}{2}} \right]}{\left( [\lambda(T)Tt] \text{Var} [W_1^{(T)}] \right)^{\frac{2+\delta/2}{2}}} \\ & \leq (\text{constant}) \limsup_{T \rightarrow \infty} \frac{\lambda(T)T^{(2+\frac{\delta}{2})H+1}\bar{F}_L(T)}{(\lambda(T)T^{2H+1}\bar{F}_L(T))^{\frac{2+\delta/2}{2}}} = (\text{constant}) \lim_{T \rightarrow \infty} (\eta(T))^{-\frac{\delta}{2}} = 0 \end{aligned}$$

by the fast growth condition (3.4). Also, since, by (5.8),

$$\lfloor \lambda(T)Tt \rfloor \text{Var} \left[ W_1^{(T)} \right] \sim \sigma_1^2 t^{H(2-\alpha_J)+1} (T^H \eta(T))^2,$$

we have by Lyapunov's Central Limit Theorem,

$$\frac{\sum_{k=1}^{\lfloor \lambda(T)Tt \rfloor} \left( W_{1,k}^{(T)} - \mathbb{E} \left[ W_1^{(T)} \right] \right)}{T^H \eta(T)} \Rightarrow \sigma_1 N \left( 0, t^{H(2-\alpha_J)+1} \right),$$

where  $N(0, t)$  is a normal random variable with mean 0 and variance  $t$ . Further, we know that  $P_1^{(T)}$  is a Poisson random variable with parameter  $\lambda(T)\hat{F}_L(Tt) \sim \lfloor \lambda(T)Tt \rfloor \rightarrow \infty$ , and therefore we have  $P_1^{(T)} / \lfloor \lambda(T)Tt \rfloor \xrightarrow{P} 1$ . Hence by Theorem 4.1.2 of Gnedenko and Korolev [1996], we have,

$$\frac{\sum_{k=1}^{P_1^{(T)}} W_{1,k}(T) - P_1^{(T)} \mathbb{E} \left[ W_1^{(T)} \right]}{T^H \eta(T)} \Rightarrow \sigma_1 N \left( 0, t^{H(2-\alpha_J)+1} \right).$$

Finally, we observe that,

$$\frac{(P_1^{(T)} - \mathbb{E}[P_1^{(T)}]) \mathbb{E} \left[ W_1^{(T)} \right]}{T^H \eta(T)} \sim \frac{(P_1^{(T)} - \mathbb{E}[P_1^{(T)}])}{\sqrt{\mathbb{E}[P_1^{(T)}]}} \mathbb{E} \left[ W_1^{(T)} \right] (T^{2H} \bar{F}_L(T))^{-\frac{1}{2}} \sqrt{t}$$

is  $o_P(1)$ , since  $E(W_1^{(T)}) \sim \mu_J$  and  $T^{2H} \bar{F}_L(T) \in RV_{H(2-\alpha_J)}$  and hence increases to  $\infty$ . Combining, we get,

$$(7.2) \quad \frac{\sum_{k=1}^{P_1^{(T)}} W_{1,k}(T) - \mathbb{E}[P_1^{(T)}] \mathbb{E} \left[ W_1^{(T)} \right]}{T \eta(T)} \stackrel{d}{=} \frac{X_1^{(T)}(t) - \mathbb{E} \left[ X_1^{(T)}(t) \right]}{T \eta(T)} \Rightarrow \sigma_1 N \left( 0, t^{H(2-\alpha_J)+1} \right).$$

For the second region determined by  $\mathcal{R}_2(T)$ , we consider equations (5.15) and (5.16) and get

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \frac{\lfloor \lambda(T) \mu_L \rfloor \mathbb{E} \left[ \left| S_2^{(T)}(Tt - \tau_2^{(T)}) - \mathbb{E} \left[ S_2^{(T)}(Tt - \tau_2^{(T)}) \right] \right|^{2+\frac{\delta}{2}} \right]}{\left( \lfloor \lambda(T) \mu_L \rfloor \text{Var} \left[ S_2^{(T)}(Tt - \tau_2^{(T)}) \right] \right)^{\frac{2+\delta/2}{2}}} \\ & \leq (\text{constant}) \limsup_{T \rightarrow \infty} \frac{\lambda(T) T^{H(2+\frac{\delta}{2})+1} \bar{F}_L(T)}{(\lambda(T) T^{2H+1} \bar{F}_L(T))^{\frac{2+\delta/2}{2}}} = (\text{constant}) \lim_{T \rightarrow \infty} (\eta(T))^{-\frac{\delta}{2}} = 0. \end{aligned}$$

Also, from (5.15), we have,

$$\lfloor \lambda(T) \mu_L \rfloor \text{Var} \left[ S_2^{(T)}(Tt - \tau_2^{(T)}) \right] \sim \sigma_2^2 t^{H(2-\alpha_J)+1} (T^H \eta(T))^2,$$

and  $P_2^{(T)}/(\lfloor \lambda(T)\mu_L \rfloor) \xrightarrow{P} 1$ . So again, using Lyapunov's Central Limit Theorem and Theorem 4.1.2 of Gnedenko and Korolev [1996], we get

$$\frac{\sum_{k=1}^{P_2^{(T)}} S_2^{(T)}(Tt - \tau_2^{(T)}) - P_2^{(T)} \mathbb{E} \left[ S_2^{(T)}(Tt - \tau_2^{(T)}) \right]}{T^H \eta(T)} \Rightarrow \sigma_2 N \left( 0, t^{H(2-\alpha_J)+1} \right).$$

To change the centering to a non-random one, observe from (5.14),

$$\begin{aligned} & \frac{P_2^{(T)} - \mathbb{E}[P_2^{(T)}]}{T^H \eta(T)} \mathbb{E} \left[ S_2^{(T)}(Tt - \tau_2^{(T)}) \right] \\ & \sim (\text{constant}) \frac{P_2^{(T)} - \mathbb{E}[P_2^{(T)}]}{\sqrt{\mathbb{E}[P_2^{(T)}]}} \frac{\mathbb{E} \left[ S_2^{(T)}(Tt - \tau_2^{(T)}) \right]}{T^{H+1} \bar{F}_L(T)} \sqrt{T \bar{F}_L(T)} = o_P(1), \end{aligned}$$

since  $H\alpha_J > 1$ . Combining, we get,

$$\begin{aligned} & \frac{\sum_{k=1}^{P_1^{(T)}} S_2^{(T)}(Tt - \tau_2^{(T)}) - \mathbb{E}[P_1^{(T)}] \mathbb{E} \left[ S_2^{(T)}(Tt - \tau_2^{(T)}) \right]}{T^H \eta(T)} \\ (7.3) \quad & \stackrel{d}{=} \frac{X_2^{(T)}(t) - \mathbb{E} \left[ X_2^{(T)}(t) \right]}{T^H \eta(T)} \Rightarrow \sigma_2 N \left( 0, t^{H(2-\alpha_J)+1} \right). \end{aligned}$$

Finally, since,  $X_1^{(T)}$  and  $X_2^{(T)}$  are independent, adding (7.2) and (7.3), we get

**Theorem 7.1.** *Under the assumptions (7) - (11) and fast growth condition (F), we have, for each  $t > 0$ ,*

$$\frac{X^{(T)}(Tt) - \mathbb{E} \left[ X^{(T)}(Tt) \right]}{T^H \eta(T)} \Rightarrow \sigma N \left( 0, t^{H(2-\alpha_J)+1} \right),$$

where  $\sigma^2 = \sigma_1^2 + \sigma_2^2 = \frac{2}{(2-\alpha_J)[H(2-\alpha_J)+1]} \mathbb{E} \left[ A_1(1)^{2-\alpha_J} \right]$ .

**7.2. Finite-dimensional convergence.** To study the finite dimensional convergence, we need the following lemma.

**Lemma 7.1.** *Suppose (7)–(11) hold and  $\phi \in RV_{l-\alpha_J}$ ,  $\alpha_J < l < 2 + \delta$ . Then as  $u \rightarrow \infty$*

$$(7.4) \quad \mathbb{E} \left[ \phi(A_1(u)) \right] \sim \mathbb{E} \left[ A_1(1)^{l-\alpha_J} \right] \phi(u^H) \in RV_{H(l-\alpha_J)}.$$

*In particular:*

(i) *If  $\alpha_J < l < 2 + \delta$  and*

$$(7.5) \quad \phi(s) = \int_0^s w^l F_J(dw) \in RV_{l-\alpha_J},$$

*then*

$$\begin{aligned} \mathbb{E} \left[ \phi(A_1(u)) \right] &= \mathbb{E} \left[ \int_0^{A_1(u)} w^l F_J(dw) \right] \sim \mathbb{E} \left[ A_1(1)^{l-\alpha_J} \right] \int_0^{u^H} w^l F_J(dw) \\ (7.6) \quad &\sim \frac{\alpha_J}{l - \alpha_J} \mathbb{E} \left[ A_1(1)^{l-\alpha_J} \right] u^{Hl} \bar{F}_J(u^H). \end{aligned}$$

(ii) If we have  $l = 2$  and

$$(7.7) \quad \phi(s) = s \int_s^\infty w F_J(dw) \in RV_{2-\alpha_J},$$

then

$$(7.8) \quad \begin{aligned} \mathbb{E}[\phi(A_1(u))] &= \mathbb{E} \left[ A_1(u) \int_{A_1(u)}^\infty w F_J(dw) \right] \sim \mathbb{E} [A_1(1)^{2-\alpha_J}] u^H \int_{u^H}^\infty w F_J(dw) \\ &\sim \frac{\alpha_J}{l - \alpha_J} \mathbb{E} [A_1(1)^{2-\alpha_J}] u^{2H} \bar{F}_J(u^H). \end{aligned}$$

Furthermore, if  $0 < l \leq \alpha_J$  and

$$(7.9) \quad \phi(s) = s^l \bar{F}_J(s) \in RV_{l-\alpha_J},$$

then (7.4) continues to hold:

$$(7.10) \quad \mathbb{E} [A_1(u)^l \bar{F}_J(A_1(u))] \sim \mathbb{E} [A_1(1)^{l-\alpha_J}] u^{Hl} \bar{F}_J(u^H).$$

*Proof.* Pick  $\rho > 0$  and write

$$\frac{\mathbb{E}[\phi(A_1(u))]}{\phi(u^H)} = \frac{\mathbb{E}[\phi(u^H A_1(1))]}{\phi(u^H)} \geq \mathbb{E} \left[ \frac{\phi(u^H A_1(1))}{\phi(u^H)} 1_{[A_1(1) \geq \rho]} \right].$$

Now for  $\alpha_J < l < 2 + \delta$ , we choose  $\varepsilon > 0$  small enough that  $l + \varepsilon < 2 + \delta$  and we have by Potter's bounds that for some constant  $c > 0$  and all large  $u$

$$\frac{\phi(u^H A_1(1))}{\phi(u^H)} 1_{[A_1(1) \geq \rho]} \leq c A_1(1)^{l-\alpha_J+\varepsilon} \in L_1.$$

So by dominated convergence, as  $u \rightarrow \infty$

$$(7.11) \quad \mathbb{E} \left[ \frac{\phi(u^H A_1(1))}{\phi(u^H)} 1_{[A_1(1) \geq \rho]} \right] \rightarrow \mathbb{E} [A_1(1)^{l-\alpha_J} 1_{[A_1(1) \geq \rho]}]$$

and therefore

$$\liminf_{u \rightarrow \infty} \frac{\mathbb{E}[\phi(A_1(u))]}{\phi(u^H)} \geq \mathbb{E} [A_1(1)^{l-\alpha_J}].$$

For the rest of the proof of (7.4) when  $l - \alpha_J > 0$ , it suffices to show

$$(7.12) \quad \lim_{u \rightarrow \infty} \mathbb{E} \left[ \frac{\phi(A_1(u))}{\phi(u^H)} 1_{[A_1(1) \leq \rho]} \right] = \mathbb{E} [A_1(1)^{l-\alpha_J} 1_{[A_1(1) \leq \rho]}].$$

Since  $l - \alpha_J > 0$ , we have uniform convergence in neighborhoods of 0 in the regular variation ratio and thus

$$\sup_{0 \leq a \leq \rho} \left| \frac{\phi(va)}{\phi(v)} - a^{l-\alpha_J} \right| =: \eta_v(\rho)$$

with  $\lim_{v \rightarrow \infty} \eta_v(\rho) = 0$ . Now

$$\begin{aligned} \left| \mathbb{E} \left[ \frac{\phi(A_1(u))}{\phi(u^H)} 1_{[A_1(1) \leq \rho]} \right] - \mathbb{E} [A_1(1)^{l-\alpha_J} 1_{[A_1(1) \leq \rho]}] \right| &\leq \mathbb{E} \left[ 1_{[A_1(1) \leq \rho]} \left| \frac{\phi(u^H A_1(1))}{\phi(u^H)} - A_1(1)^{l-\alpha_J} \right| \right] \\ &\leq \eta_v(\rho) \mathbb{P}[A_1(1) \leq \rho] \rightarrow 0, \end{aligned}$$

as  $u \rightarrow \infty$ .



Now consider the proof of (7.10) when  $0 < l \leq \alpha_J$  and (7.9) hold. Set  $v = u^H$ . We have, by independence of  $J_1$  and  $A_1$

$$\begin{aligned} & \mathbb{E} \left[ \frac{(A_1(1))^l \bar{F}_J(v A_1(1)) \mathbf{1}_{[A_1(1) < \rho]}}{\bar{F}_J(v)} \right] \\ &= \mathbb{E} \left[ \frac{(A_1(1))^l \mathbf{1}_{[J_1 > v A_1(1)]} \mathbf{1}_{[A_1(1) < \rho]} \mathbf{1}_{[J_1 \leq v \rho]}}{\bar{F}_J(v)} \right] + \mathbb{E} \left[ \frac{(A_1(1))^l \mathbf{1}_{[J_1 > v A_1(1)]} \mathbf{1}_{[A_1(1) < \rho]} \mathbf{1}_{[J_1 > v \rho]}}{\bar{F}_J(v)} \right] \\ &\leq \mathbb{E} \left[ \frac{J_1^l \mathbf{1}_{[A_1(1)^{-\alpha_J} > (J_1/v)^{-\alpha_J}]} \mathbf{1}_{[J_1 \leq v \rho]}}{v^l \bar{F}_J(v)} \right] + \mathbb{E} \left[ \frac{\rho^l \mathbf{1}_{[A_1(1)^{-\alpha_J} > \rho^{-\alpha_J}]} \mathbf{1}_{[J_1 > v \rho]}}{\bar{F}_J(v)} \right] \end{aligned}$$

and applying Chebyshev's inequality, this is bounded by

$$\begin{aligned} &\leq \frac{\mathbb{E} [J_1^{l+\alpha_J} \mathbf{1}_{[J_1 \leq v \rho]}]}{v^{l+\alpha_J} \bar{F}_J(v)} \mathbb{E} [A_1(1)^{-\alpha_J}] + \rho^{l+\alpha_J} \frac{\bar{F}_J(\rho v)}{\bar{F}_J(v)} \mathbb{E} [A_1(1)^{-\alpha_J}], \\ &\rightarrow \frac{\alpha_J}{l} \rho^l \mathbb{E} [A_1(1)^{-\alpha_J}] + \rho^l \mathbb{E} [A_1(1)^{-\alpha_J}], \end{aligned}$$

by by Karamata's theorem as  $v \rightarrow \infty$ . Thus, from (7.11)

$$\begin{aligned} &\limsup_{v \rightarrow \infty} \mathbb{E} \left[ \frac{(v A_1(1))^l \bar{F}_J(v A_1(1))}{v^l \bar{F}_J(v)} \right] \\ &\leq \limsup_{v \rightarrow \infty} \mathbb{E} \left[ \frac{A_1(1)^l \bar{F}_J(v A_1(1)) \mathbf{1}_{[A_1(1) \geq \rho]}}{\bar{F}_J(v)} \right] + \limsup_{v \rightarrow \infty} \mathbb{E} \left[ \frac{A_1(1)^l \bar{F}_J(v A_1(1)) \mathbf{1}_{[A_1(1) < \rho]}}{\bar{F}_J(v)} \right] \\ &\leq \mathbb{E} [A_1(1)^{l-\alpha_J} \mathbf{1}_{[A_1(1) \geq \rho]}] + \frac{\alpha_J}{l} \rho^l \mathbb{E} [A_1(1)^{-\alpha_J}] + \rho^l \mathbb{E} [A_1(1)^{-\alpha_J}]. \end{aligned}$$

Letting  $\rho \rightarrow 0$ , we have

$$(7.13) \quad \limsup_{v \rightarrow \infty} \frac{\mathbb{E} [(v A_1(1))^l \bar{F}_J(v A_1(1))]}{v^l \bar{F}_J(v)} \leq \mathbb{E} [A_1(1)^{l-\alpha_J}].$$

This gives the desired result.  $\square$

For the finite dimensional convergence, we consider only two-dimensional convergence to make notation simpler. The general case follow similarly. Let  $N^{(T)}(t) = \sum_k \varepsilon_{\Gamma_k^{(T)}}([0, t])$  be the counting process corresponding to the initiation times  $\{\Gamma_k^{(T)}\}$  in  $[0, t]$ . Observe that

$$X^{(T)}(t) = \sum_{k=1}^{N^{(T)}(t)} A_k \left( t - \Gamma_k^{(T)} \right) \wedge J_k.$$

Recall that  $\sum_k \varepsilon_{(\Gamma_k^{(T)}, A_k, J_k)}$  is a Poisson random measure. Now fix  $0 < s < t$ . Break the previous sum defining  $X^{(T)}$  into the independent pieces corresponding to  $\sum_{0 \leq \Gamma_k^{(T)} \leq T s} + \sum_{T s < \Gamma_k^{(T)} \leq T t}$ . Then, using stationary, independent increments of  $N^{(T)}$  and the order statistic property of a Poisson

process, we have

$$\begin{aligned}
 \begin{pmatrix} X^{(T)}(Ts) \\ X^{(T)}(Tt) \end{pmatrix} &= \begin{pmatrix} \sum_{k=1}^{N^{(T)}(Ts)} A_k \left( Ts - \Gamma_k^{(T)} \right) \wedge J_k \\ \sum_{k=1}^{N^{(T)}(Tt)} A_k \left( Tt - \Gamma_k^{(T)} \right) \wedge J_k \end{pmatrix} \\
 (7.14) \quad &\stackrel{d}{=} \sum_{k=1}^{N_1^{(T)}(Ts)} \begin{pmatrix} A_{k,1}(TsU_{k,1}) \wedge J_{k,1} \\ A_{k,1}(T(t-s) + TsU_{k,1}) \wedge J_{k,1} \end{pmatrix} + \sum_{k=1}^{N_2^{(T)}(T(t-s))} \begin{pmatrix} 0 \\ A_{k,2}(T(t-s)U_{k,2}) \wedge J_{k,2} \end{pmatrix},
 \end{aligned}$$

with  $\{N_i^{(T)}, i \geq 1\}$  being i.i.d. copies of  $N^{(T)}$ ,  $U_{k,i}$  being i.i.d. copies of Uniform(0,1) random variables,  $J_{k,i}$  and  $A_{k,i}$  being i.i.d. copies of  $J_1$  and  $A_1$  respectively. A typical summand in the first sum represents accumulation by times  $Ts$  and  $Tt$  from a transmission initiated at  $Ts(1 - U_{k,1})$ , while the summand in the second component of the second sum represents accumulation from transmissions initiated in  $(Ts, Tt]$ . Now, the second sum on the right side of (7.14) is independent of the first sum and the second coordinate has the same distribution as  $X^{(T)}(T(t-s))$ , and hence has a Gaussian limit after centering by the mean and scaling by  $T^H \eta(T)$ . So we need to study the first term only. Let us define

$$Y_{k,s,t}^{(T)} = A_k(Tt + TsU_1) \wedge J_k.$$

Then

$$\begin{aligned}
 \mathbb{E} \left[ \left( Y_{1,s,t}^{(T)} \right)^l \right] &= \int_{u=0}^1 \int_{w=0}^\infty \mathbb{E} \left[ (A_1(Tt + Tsu) \wedge w)^l \right] F_J(dw) du \\
 (7.15) \quad &= \int_{u=0}^1 \mathbb{E} \left[ A_1(Tt + Tsu)^l \bar{F}_J(A_1(Tt + Tsu)) \right] du + \int_{u=0}^1 \mathbb{E} \left[ \int_{w=0}^{A_1(Tt+Tsu)} w^l F_J(dw) \right] du.
 \end{aligned}$$

Now, observe that, for  $1 \leq l < 2 + \delta$ , by substitution of variable,

$$\begin{aligned}
 &\int_{u=0}^1 \mathbb{E} \left[ A_1(Tt + Tsu)^l \bar{F}_J(A_1(Tt + Tsu)) \right] du \\
 &= \frac{1}{Ts} \int_{u=Tt}^{T(s+t)} \mathbb{E} \left[ A_1(u)^l \bar{F}_J(A_1(u)) \right] du \\
 &= \frac{1}{Ts} \int_{u=Tt}^{T(s+t)} \mathbb{E} \left[ (u^H A_1(1))^l \bar{F}_J(u^H A_1(1)) \right] du \\
 (7.16) \quad &\sim \frac{1}{H(l - \alpha_J) + 1} \frac{(s+t)^{H(l - \alpha_J) + 1} - t^{H(l - \alpha_J) + 1}}{s} \mathbb{E} \left[ A_1(1)^{l - \alpha_J} \right] T^{Hl} \bar{F}_J(T^H),
 \end{aligned}$$

using Karamata's theorem and Lemma 7.1, since, by assumption (10b),  $H(l - \alpha_J) + 1 \geq H(1 - \alpha_J) + 1 > 0$ . For the second term of (7.15), observe that, again changing variables  $u' = us + t$ , we

get

$$(7.17) \quad \int_{u=0}^1 \mathbb{E} \left[ \int_{w=0}^{A_1(Tt+Tsu)} w^l F_J(dw) \right] du = \frac{1}{s} \int_{u=t}^{t+s} \mathbb{E} \left[ \int_{w=0}^{A_1(Tu)} w^l F_J(dw) \right] du,$$

$$\sim \begin{cases} \mathbb{E} [J_1^l], & \text{for } 0 < l < \alpha_J \\ \frac{\alpha_J}{l-\alpha_J} \frac{(s+t)^{H(l-\alpha_J)+1} - t^{H(l-\alpha_J)+1}}{[H(l-\alpha_J)+1]s} \mathbb{E} [A_1(1)^{l-\alpha_J}] T^{Hl} \bar{F}_L(T), & \text{for } \alpha_J < l < 2 + \delta \end{cases}$$

where we use Monotone Convergence Theorem in the first case and Karamata's theorem and Lemma 7.1 in the other. We also used  $\bar{F}_L(T) \sim \bar{F}_J(T^H)$ . Hence, combining (7.16) and (7.17), we have from (7.15),

$$\mathbb{E} \left[ \left( Y_{1,s,t}^{(T)} \right)^l \right] \rightarrow \mathbb{E} [J_1^l], \quad \text{for } 1 \leq l < \alpha_J$$

$$\mathbb{E} \left[ \left( Y_{1,s,t}^{(T)} \right)^l \right] = O(T^{Hl} \bar{F}_L(T)), \text{ for } \alpha_J < l < 2 + \delta.$$

Thus, we have, as  $T \rightarrow \infty$ ,

$$(7.18) \quad \frac{\text{Var} [Y_{1,s,t}^{(T)}]}{T^{2H} \bar{F}_L(T)} \rightarrow \frac{2}{2 - \alpha_J} \frac{(s+t)^{H(l-\alpha_J)+1} - t^{H(l-\alpha_J)+1}}{[H(2 - \alpha_J) + 1]s} \mathbb{E} [A_1(1)^{2-\alpha_J}]$$

and

$$(7.19) \quad \mathbb{E} \left[ \left| Y_{1,s,t}^{(T)} - \mathbb{E} [Y_{1,s,t}^{(T)}] \right|^{2+\delta/2} \right] = O(T^{H(2+\delta/2)} \bar{F}_L(T)).$$

Next we study the covariance between  $Y_{1,s,0}^{(T)}$  and  $Y_{1,s,t-s}^{(T)}$ . Observe that by decomposing

$$[0, \infty) = [0, A_1(Tsu)] \cup (A_1(Tsu), A_1(Tsu + T(t-s))] \cup (A_1(Tsu + T(t-s)), \infty),$$

we get

$$\begin{aligned} \mathbb{E} \left[ \frac{Y_{1,s,0}^{(T)} Y_{1,s,t-s}^{(T)}}{T^{2H} \bar{F}_J(T^H)} \right] &= \int_{u=0}^1 \int_{w=0}^\infty \frac{\mathbb{E} [(A_1(Tsu) \wedge w) (A_1(T(t-s) + Tsu) \wedge w)] F_J(dw) du}{T^{2H} \bar{F}_J(T^H)} \\ &= \int_{u=0}^1 \mathbb{E} \left[ \frac{\int_{w=0}^{A_1(Tsu)} w^2 F_J(dw)}{T^{2H} \bar{F}_J(T^H)} \right] du + \int_{u=0}^1 \mathbb{E} \left[ \frac{A_1(Tsu) \int_{w=A_1(Tsu)}^\infty w F_J(dw)}{T^{2H} \bar{F}_J(T^H)} \right] du \\ &\quad - \int_{u=0}^1 \mathbb{E} \left[ \frac{A_1(Tsu) \int_{w=A_1(T(t-s)+Tsu)}^\infty w F_J(dw)}{T^{2H} \bar{F}_J(T^H)} \right] du \\ &\quad + \int_{u=0}^1 \mathbb{E} \left[ \frac{A_1(Tsu) A_1(T(t-s) + Tsu) \bar{F}_J(A_1(T(t-s) + Tsu))}{T^{2H} \bar{F}_J(T^H)} \right] du. \end{aligned}$$

We treat each of the terms above separately. The first term gives us

$$\int_{u=0}^1 \mathbb{E} \left[ \frac{\int_{w=0}^{A_1(Tsu)} w^2 F_J(dw)}{T^{2H} \bar{F}_J(T^H)} \right] du = \int_{u=0}^{Ts} \mathbb{E} \left[ \frac{\int_{w=0}^{A_1(u)} w^2 F_J(dw)}{Ts \cdot T^{2H} \bar{F}_J(T^H)} \right] du$$

$$\rightarrow \frac{\alpha_J}{(2 - \alpha_J)[H(2 - \alpha_J) + 1]} \mathbb{E}[A_1(1)^{2-\alpha_J}] s^{H(2-\alpha_J)}$$

using Lemma 7.1. The second term gives us, also using Lemma 7.1,

$$\begin{aligned} \int_{u=0}^1 \mathbb{E} \left[ \frac{A_1(Tsu) \int_{w=A_1(Tsu)}^{\infty} w F_J(dw)}{T^{2H} \bar{F}_J(T^H)} \right] du &= \int_{u=0}^{Ts} \mathbb{E} \left[ \frac{A_1(u) \int_{w=A_1(u)}^{\infty} w F_J(dw)}{T^{2H} \bar{F}_J(T^H)} \right] du \\ &\rightarrow \frac{\alpha_J}{(\alpha_J - 1)[H(2 - \alpha_J) + 1]} \mathbb{E}[A_1(1)^{2-\alpha_J}] s^{H(2-\alpha_J)} \end{aligned}$$

For the third term, observe that

$$\frac{A_1(Tsu) \int_{w=A_1(T(t-s)+Tsu)}^{\infty} w F_J(dw)}{T^{2H} \bar{F}_J(T^H)} \leq \frac{A_1(Tsu) \int_{w=A_1(Tsu)}^{\infty} w F_J(dw)}{T^{2H} \bar{F}_J(T^H)}$$

and hence from Dominated Convergence theorem, using the analysis of the second term,

$$\begin{aligned} \int_{u=0}^1 \mathbb{E} \left[ \frac{A_1(Tsu) \int_{w=A_1(T(t-s)+Tsu)}^{\infty} w F_J(dw)}{T^{2H} \bar{F}_J(T^H)} \right] du &= \int_{u=0}^1 \mathbb{E} \left[ \frac{A_1(su) \int_{w=T^H A_1((t-s)+su)}^{\infty} w F_J(dw)}{T^H \bar{F}_J(T^H)} \right] du \\ &\rightarrow \frac{\alpha_J}{\alpha_J - 1} \int_{u=0}^1 \mathbb{E} [A_1(su) A_1((t-s) + su)^{1-\alpha_J}] du. \end{aligned}$$

Finally, observe that, the fourth term is

$$\begin{aligned} \int_{u=0}^1 \mathbb{E} \left[ \frac{A_1(Tsu) A_1(T(t-s) + Tsu) \bar{F}_J(A_1(T(t-s) + Tsu))}{T^{2H} \bar{F}_J(T^H)} \right] du \\ = \int_{u=0}^1 \mathbb{E} \left[ \frac{A_1(su) A_1(t-s + su) \bar{F}_J(T^H A_1(t-s + su))}{\bar{F}_J(T^H)} \right] du. \end{aligned}$$

Now,  $A_1(su) A_1(t-s + su) \bar{F}_J(T^H A_1(t-s + su)) / \bar{F}_J(T^H)$  converges to  $A_1(su) A_1(t-s + su)^{1-\alpha_J}$  almost surely and is bounded by  $A_1(t-s + su)^2 \bar{F}_J(T^H A_1(t-s + su)) / \bar{F}_J(T^H)$ . Also

$$\begin{aligned} \int_{u=0}^1 \mathbb{E} \left[ \frac{A_1(t-s + su)^2 \bar{F}_J(T^H A_1(t-s + su))}{\bar{F}_J(T^H)} \right] \\ = \int_{u=t-s}^t \mathbb{E} \left[ \frac{A_1(Tu)^2 \bar{F}_J(A_1(Tu))}{s \cdot T^{2H} \bar{F}_J(T^H)} \right] = \int_{u=T(t-s)}^{Tt} \mathbb{E} \left[ \frac{A_1(u)^2 \bar{F}_J(A_1(u))}{Ts \cdot T^{2H} \bar{F}_J(T^H)} \right] \\ \rightarrow \frac{1}{H(2 - \alpha_J) + 1} \frac{t^{H(2-\alpha_J)+1} - (t-s)^{H(2-\alpha_J)+1}}{s} \mathbb{E}[A_1(1)^{2-\alpha_J}]. \end{aligned}$$

Hence, the fourth term converges to  $\int_{u=0}^1 \mathbb{E} [A_1(su) A_1(t-s + su)^{1-\alpha_J}] du$ , by the Dominated Convergence Theorem. Putting all the terms together, we have,

$$\frac{\text{Cov}(Y_{1,s,0}^{(T)}, Y_{1,s,t-s}^{(T)})}{T^{2H} \bar{F}_J(T^H)} \sim \mathbb{E} \left[ \frac{Y_{1,s,0}^{(T)} Y_{1,s,t-s}^{(T)}}{T^{2H} \bar{F}_J(T^H)} \right]$$

$$\begin{aligned}
& \sim \frac{\alpha_J \mathbb{E}[A_1(1)^{2-\alpha_J}] s^{H(2-\alpha_J)}}{(2-\alpha_J)[H(2-\alpha_J)+1]} + \frac{\alpha_J \mathbb{E}[A_1(1)^{2-\alpha_J}] s^{H(2-\alpha_J)}}{(\alpha_J-1)[H(2-\alpha_J)+1]} \\
& - \frac{\alpha_J}{\alpha_J-1} \int_{u=0}^1 \mathbb{E}[A_1(su)A_1((t-s)+su)^{1-\alpha_J}] du + \int_{u=0}^1 \mathbb{E}[A_1(su)A_1(t-s+su)^{1-\alpha_J}] du \\
(7.20) \quad & = \frac{\alpha_J \mathbb{E}[A_1(1)^{2-\alpha_J}] s^{H(2-\alpha_J)}}{(\alpha_J-1)(2-\alpha_J)[H(2-\alpha_J)+1]} - \frac{1}{\alpha_J-1} \int_{u=0}^1 \mathbb{E}[A_1(su)A_1((t-s)+su)^{1-\alpha_J}] du
\end{aligned}$$

Thus, if  $\mathbf{Y}_{k,s,t}^{(T)} = (Y_{k,s,0}^{(T)}, Y_{k,s,t-s}^{(T)})^T$  and  $\text{Var}[\mathbf{Y}_{s,t}^{(T)}] = \Sigma_{s,t}^{(T)}$ , using (7.18) and (7.20), we have,  $\Sigma_{s,t}^{(T)} / (T^{2H} \bar{F}_J(T))$  converges to a positive-definite covariance matrix  $\Sigma_{s,t}$ , say.

Then, for any 2-dimensional vector  $\mathbf{a} = (a_1, a_2)^T$ , we have

$$\lim_{T \rightarrow \infty} \frac{\text{Var}[\mathbf{a}^T \mathbf{Y}_{1,s,t}^{(T)}]}{T^{2H} \bar{F}_J(T^H)} = \lim_{T \rightarrow \infty} \frac{\mathbf{a}^T \Sigma_{s,t}^{(T)} \mathbf{a}}{T^{2H} \bar{F}_J(T^H)} = \mathbf{a}^T \Sigma_{s,t} \mathbf{a},$$

which is non-zero. On the other hand, by Minkowski's inequality, we have

$$\mathbb{E} \left[ \left| \mathbf{a}^T \mathbf{Y}_{1,s,t}^{(T)} - \mathbb{E}[\mathbf{a}^T \mathbf{Y}_{1,s,t}^{(T)}] \right|^{2+\delta/2} \right] = O \left( T^{H(2+\delta/2)} \bar{F}_L(T) \right).$$

So  $\mathbf{a}^T \mathbf{Y}_{1,s,t}^{(T)}$  satisfies Lyapunov's condition

$$\frac{[\lambda(T)T] \mathbb{E} \left[ \left| \mathbf{a}^T \mathbf{Y}_{1,s,t}^{(T)} - \mathbb{E}[\mathbf{a}^T \mathbf{Y}_{1,s,t}^{(T)}] \right|^{2+\frac{\delta}{2}} \right]}{[T^H \eta(T)]^{2+\frac{\delta}{2}}} \leq (\text{constant}) \limsup_{T \rightarrow \infty} (\eta(T))^{-\frac{\delta}{2}} = 0.$$

Hence, using Lyapunov's Central Limit Theorem and Cramer-Wold device, we have that

$$\sum_{k=1}^{\lfloor \lambda(T)T \rfloor} \frac{(\mathbf{Y}_{k,s,t}^{(T)} - \mathbb{E}[\mathbf{Y}_{1,s,t}^{(T)}])}{T^H \eta(T)}$$

converges weakly to a normal random vector with zero mean and covariance matrix  $\Sigma_{s,t}$ . Also, we have

$$\frac{N_1^{(T)}(Ts)}{\lfloor \lambda(T)T \rfloor} \xrightarrow{P} s.$$

Hence, by using Theorem 4.1.2 of Gnedenko and Korolev [1996], we have

$$\frac{\sum_{k=1}^{N_1^{(T)}(Ts)} \mathbf{Y}_{k,s,t}^{(T)} - \mathbb{E} \left[ \sum_{k=1}^{N_1^{(T)}(Ts)} \mathbf{Y}_{s,t}^{(T)} \right]}{T^H \eta(T)}$$

converges weakly to a normal random vector with zero mean and covariance matrix  $\sqrt{s} \Sigma_{s,t}$ . Note that we can change the centering as in the one-dimensional case, since the changes are probabilistically negligible coordinate by coordinate. Recall from (7.14), the two dimensional distribution of  $X^{(T)}$  is sum of two independent terms, both of which are now shown to be asymptotically Gaussian on centering by mean and scaling by  $T^H \eta(T)$ . Hence, we have

$$\frac{1}{T^H \eta(T)} \left\{ \begin{pmatrix} X^{(T)}(Ts) \\ X^{(T)}(Tt) \end{pmatrix} - \mathbb{E} \left[ \begin{pmatrix} X^{(T)}(Ts) \\ X^{(T)}(Tt) \end{pmatrix} \right] \right\}$$

converges to a zero mean Gaussian process  $\sigma G(\cdot)$  in the sense of the convergence of finite dimensional distribution.

To understand the Gaussian process better, we look at its second order properties. From the one dimensional convergence given in Theorem 7.1, we know that  $\text{Var}[G(t)] = t^{H(2-\alpha_J)+1}$ . Also, from (7.14), it is clear that only the first term on the right side contributes to the asymptotic covariance  $\text{Cov}[G(s), G(t)]$  and so from the limiting covariance matrix of  $\sum_{k=1}^{N_1^{(T)}(Ts)} \mathbf{Y}_{k,s,t}^{(T)} / (T^H \eta(T))$  and (7.20), we have  $\text{Cov}[G(s), G(t)]$  is homogeneous of order  $H(2 - \alpha_J) + 1$ . So  $G$  is a self-similar process of index  $[H(2 - \alpha_J) + 1]/2$ . We summarize this in the following theorem:

**Theorem 7.2.** *Under the assumptions (7) - (11) and the fast growth condition (F), we have*

$$\frac{X^{(T)}(T\cdot) - \mathbb{E}[X^{(T)}(T\cdot)]}{T^H \eta(T)} \xrightarrow{\text{fidi}} \sigma G(\cdot),$$

where  $G$  is a zero mean self-similar Gaussian process with self-similarity index  $\frac{H(2-\alpha_J)+1}{2}$ .

## 8. CONCLUSION

In this paper, we have proposed a family of models, each of which exhibits multifractal behavior at small time scales and we obtain two different types of limiting behaviors for the process  $X^{(T)}(\cdot)$  for large time scale depending on the growth condition. Under the slow growth condition,  $X^{(T)}(\cdot)$  when scaled by  $b_J(\lambda(T)T)$  has a centered limit which is a right-skewed stable Lévy motion of self-similarity index  $1/\alpha_J$ . Under the fast growth condition,  $X^{(T)}(\cdot)$  needs to be scaled by  $\sqrt{\lambda(T)T^{2H+1}\bar{F}_L(T)}$  and the centered limit is a Gaussian process of self-similarity parameter  $[H(2 - \alpha_J) + 1]/2$ . Interestingly, both the self-similarity indices have a range  $(\frac{1}{2}, 1)$  using assumptions (9) and (10c) and  $1/\alpha_J < [H(2 - \alpha_J) + 1]/2$  using the conditions  $\alpha_J < 2$  and  $H\alpha_J > 1$ . Yet, the limiting behavior in the slow growth case fails to capture the effect of the individual input processes  $A_k$ , as only the terms corresponding to connections, which have ended transmissions, contribute towards the limit. The limit in the fast growth case depends on  $H$  and hence can lead to models with richer parametrization. Further we note that, the limit obtained under the fast growth condition does not have stationary increments. If the limit had stationary increment besides being Gaussian and self-similar, we could consider the limit to be fractional Brownian motion, since the index of self-similarity is in the interval  $(\frac{1}{2}, 1)$ . A possible approach towards obtaining a limit with stationary increments is to consider a stationary version of the process  $X^{(T)}$ , as has been done in the work of Mikosch et al. [2002], by looking at the contributions from time 0, where the connections may be going back in the past.

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